"Napier based his explanations upon two considerations: (1) the geometrico-mechanical concept of flowing points, (2) the relations which exist between arithmetic and geometric series. . . . Napier lets the point $g$ move along the definite line $TS$ with a diminishing velocity such that its velocity at $T$ is to that at $d$, as the distance $TS$ is to the distance $dS$. At the same time Napier lets a point $a$ move along the line $bc$ (which is of indefinite length) with a uniform velocity which is the same as the initial velocity of the point $g$. If the two points start to move at the same moment, and if $g$ is at $d$ when $a$ is at $c$, then the length $bc$ is defined as the logarithm of $dS$. Napier constructed tables for trigonometric computation. With that end in view he lets $TS$ stand for the *radius*, assigning to it the value $10^7$, while $dS$ stands for a given *sine*. At that time trigonometric functions were not thought of strictly as *ratios*."

Then I give Napier's kinematical definition *in his own words;* it is the very definition in the *Constructio*, § 26, which Professor Carslaw says in the above communication that I ignore. What more could I do to emphasize this kinematical definition?

As an easy deduction from this kinematical definition is the one-one correspondence between arithmetic progressions and geometric progressions used by Napier in his computations. The part of Professor Carslaw's criticism which I consider valid is that Nap. log $(10^7 - 1) = 1.00000005$ exhibits "Table I" in Napier's *Constructio* somewhat more closely than does Nap. log $(10^7 - 1) = 1$. This I admitted in my former reply.

COLORADO SPRINGS,
June, 1916.

## II. RELATING TO THE QUADRATIC FACTORS OF A POLYNOMIAL.

BY O. E. GLENN, University of Pennsylvania.

The process of finding by trial the rational roots of an equation $f(x) = 0$ with integral coefficients, depending upon the resolution of the last coefficient into its prime factors $s_1, s_2, \cdots$ and the first coefficient into its prime factors $r_1, r_2, \cdots$, and testing by division the possible factors $x - s/r$, is given as an isolated method in books on elementary theory of equations, with no suggestion that it is capable of extensions. As an incident in a published paper[1] on Degenerate Curves I have shown, however, that this process is a particular manifestation of a general one in which, by a finite number of arithmetical trials, one can determine any polynomial $g(x)$ with integral coefficients which is a factor of $f(x)$. As applied to quadratic factors of $f(x)$, and especially when $f(x)$ is a quartic, the extended method is very useful.

By multiplying the roots of $f(x) = 0$ by a properly chosen integer we can assume it in the form

$$f(x) = x^n + p_1x^{n-1} + \cdots + p_n = 0,$$

where the $p$'s are integers. Let a quadratic factor of $f(x)$, with integral coefficients, be $x^2 - \xi x - \eta$. If the roots of $f(x) = 0$ are $-x_1, -x_2, \cdots, -x_n$, we may assume $-\xi = x_1 + x_2$.

[1] *Amer. Journal of Math.*, vol. 32 (1910), p. 79.

We can then prove the following result, under the hypotheses:
*The integer $\xi$ must be a prime factor of the integer*

$$
\begin{aligned}
P_n = (x_1 + x_2)(x_1 + x_3) \cdots & (x_1 + x_n) \\
(x_2 + x_3) \cdots & (x_2 + x_n) \\
\cdots\cdots & \cdots \\
& (x_{n-1} + x_n),
\end{aligned}
$$

*or a product of prime factors of $P_n$.*

The number $P_n$ is an integer because it is a symmetric function of the roots and hence is rational and integral in the integers $p_1, p_2, \cdots, p_n$. The fact that $\xi$ divides $P_n$ *algebraically* does not, however, prove the proposition stated. We prove it as follows: Let the equation whose roots are the $(n) = \frac{1}{2}n(n-1)$ numbers

$$-(x_i + x_j) \qquad (i < j)$$

be

(1) $$\xi^{(n)} + \varphi_1 \xi^{(n)-1} + \varphi_2 \xi^{(n)-2} + \cdots + \varphi_{(n)} = 0.$$

Then all of the functions $\varphi_i$, being symmetric in the roots, $x_j$, are integers, and $\varphi_{(n)} = P_n$. Now $\xi$, an integer, satisfies this equation, and is a factor of all terms up to the last. Hence $\xi$ is a prime factor of $P_n$ or a product of its prime factors. Likewise $\eta$ is a factor of $p_n$.

It is now evident that in order to discover any quadratic factor of the required type of $f(x)$ we have only to resolve $P_n$ and $p_n$ into their prime factors and, with a factor of $P_n$ as $\xi$, and a factor of $p_n$ as $\eta$, test the resulting quadratic for divisibility into $f(x)$.

In the case of a quartic polynomial $f(x) = x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4$, we find

$$P_4 = p_1 p_2 p_3 - p_1^2 p_4 - p_3^2.$$

If $P_4$ or $p_4$ contain a large number of prime factors the number of trial divisions required to apply the method may be reduced by making use of the formulas for upper limits to the roots. If $L \not< 1$ is a superior limit to the absolute values of the roots, then $2L$ is a superior limit to $|\xi|$ and $L^2$ a superior limit to $|\eta|$. It is important to note, as well, that in the case of the quartic the $\xi$'s of the two quadratic factors, as $\xi_1, \xi_2$, must satisfy the condition $\xi_1 + \xi_2 = -p_1$. These conditions often render the number of tests required to resolve a quartic as small as the number required in case of the linear factors.

For illustration consider the quartic

$$f(x) = x^4 - 6x^3 + 3x^2 + 22x - 6 = 0,$$

whose solution, in condensed form, under the theory where the reducing cubic has a commensurable root, is given on page 243 in Volume I of Burnside and Panton's Theory of Equations (sixth edition).

We have

$$P_4 = -2^3.83, \qquad p_4 = -2.3,$$

and 6 is a superior limit to the roots. Hence

$$\xi = \pm 1, \pm 2, \pm 4, \pm 8; \qquad \eta = \pm 1, \pm 2, \pm 3, \pm 6.$$

But $\xi_1 + \xi_2 = 6$, hence the pair $(\xi_1, \xi_2)$ is either $(-2, 8)$ or $(2, 4)$, *i. e.*, $\xi = 2$ or $-2$. The maximum number of unsuccessful trials is 15. With $\xi = 2$, $\eta = 6$, we have, employing an obvious extension of synthetic division,

$$\begin{array}{llllll}
\underline{1} & -\ 6 & +\ 3 & +\ 22 & -\ 6 & \underline{|\ 2+6} \\
\bar{1} & \phantom{-}\ \underline{2} & \phantom{+}\ 6 & & & \\
 & -\ \bar{4} & -\ \underline{8} & -\ 24 & & \\
 & & \phantom{-}\ \bar{1} & \phantom{-}\ 2 & \phantom{-}\ 6 & \\
 & & & \overline{\phantom{-}\ 0} & \overline{\phantom{-}\ 0} &
\end{array}$$

Thus the remainder $ax + b$ is zero and

$$f(x) = (x^2 - 2x - 6)(x^2 - 4x + 1).$$

Further illustrations of quartics with irrational roots follow.

Let

$$f(x) = x^4 - x^3 - 6x^2 + 5x + 3 = 0.$$

Then

$$P_4 = 2,\ p_4 = 3; \quad \xi = \pm 1, \pm 2; \quad \eta = \pm 1, \pm 3.$$

The only pair $(\xi_1, \xi_2)$ for which $\xi_1 + \xi_2 = 1$ is $(-1, 2)$. Hence the maximum number of unsuccessful trials is three. We find

$$f(x) = (x^2 + x - 3)(x^2 - 2x - 1).$$

*Exercise* 1.—Show that the equation

$$x^4 + 4x^3 - 4x^2 - 17x + 10 = 0$$

can be resolved by less than nine trial divisions, and that the roots are $\frac{1}{2}(-1 \pm \sqrt{21})$, $\frac{1}{2}(-3 \pm \sqrt{17})$.

*Exercise* 2.—Solve the equation

$$x^4 + 4x^3 + 4x^2 - 16x + 28 = 0.$$

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## NOTES AND NEWS.

SEND ALL COMMUNICATIONS TO D. A. ROTHROCK, Indiana University.

The Secretary of the ASSOCIATION, Professor W. D. CAIRNS, is on leave of absence from Oberlin College during the present academic year. Until further notice he will be in residence at the University of Chicago and his official address will be **5465 Greenwood Avenue, Chicago, Illinois.**